Crystallizing Mathematics Concepts

How Humans Learn to Think Mathematically: Exploring The Three Worlds of Mathematics

By David Tall, Cambridge University Press, 2013, 484 pages, Paperback $39.99

The first glimpse of my new granddaughter turned my thoughts to what could be going through her mind. Just hours before finishing David Tall's *How Humans Learn to Think Mathematically* she came through my daughter's birth canal. Although she had been hearing sounds for some time, she sees daylight for the first time. To her, the world starts with a blank slate. At least we think it does. Is she wondering what this new light is all about? But what is the biochemical mechanism of wonder? How could she think at all without a processing structure to frame her wondering? In just a few weeks she will be engaging with hints that the world is bigger than her mother, and in a few months she will have built the scaffold for understanding language and mathematics.

One cannot help but wonder if Plato's Meno got it right. Did Meno's slave boy actually know the Pythagorean theorem in a previous life? Do we learn new things as we grow, or do we just recall things from a previous life? Indeed, Tall tells us on page 117 that mathematical thinking starts from "facilities set before birth in our genes and develops through successive experiences where new situations are interpreted using knowledge structures based on experiences that the individual has met before."

How do we learn to think mathematically? This is the question David Tall asks in a book that encapsulates forty years of his research on the subject, a book for the general reader as well as for mathematics education theorists. In it we learn about how the newborn puts together inklings from play actions with math objects and language as a foundation for sequential heights of conception. It starts with actions and thoughts through counting and measuring to operational symbols and onto our mathematical understanding of number. That's the start as well as the hook.

Tall's challenge is to put his mathematics education research into one book. Others have tried. There are many books that have bravely tried to tell us how we think mathematically. Jacques Hadamard's *Psychology of Invention in the Mathematics Field*, Henri Poincaré' *The Foundations of Science*, Jean Piaget's *The Language and Thought of the Child* and Lakoff and Núñez's *Where Mathematics Comes From* are the best known classics. But none of these are exposés on authentic modern scientific research. Hadamard and Poincaré both tried to recall how they themselves thought about mathematics, and they were not talking about educational development. Piaget, on the other hand was a child development researcher who came to the conclusion that intelligence develops in stages and progresses from one stage to the next only after successful completion. But Piaget did his work in the first half of the 20th century when the methods, funds, and equipment for child development research were limited. The second half of the twentieth century enjoyed far more activity in the field, accelerated public and political interest, easier means of data collection, and more refined statistical methods for studying how humans think. The second half of the twentieth century saw a plethora of research into mathematical thinking, and Tall's book informs us of almost all of the best research in the field. More recently, with PET and fMRI imaging, researchers have been studying brain activity and collecting some ideas for future probes into how humans think about mathematics. I am delighted to see complimentary connections between this book and Enlightening Symbols. My book is mostly about history the historical development of symbols and the communication of mathematical ideas across cultures. How Mathematicians Learn to Think Mathematically is mostly about the cognitive development of mathematics, from childhood to adulthood.

Like Piaget and many other mathematics education researchers, David Tall sees intellectual maturity through stages. He begins by telling us how the young child processes properties of objects and its associated language of space, shape, and number. That process starts at counting and moves to concepts such as number. Somewhere along the learning path categorization comes to feed new concepts into the symbolic world. We learn that, through practical experiences, such processing matures to a conceptualizing of actions compressed into symbols that lead to an understanding of arithmetic followed by algebraic operations for problem solving, and eventually to logical deduction. The last stage is an intermediate mathematical maturity starting from the theoretical definitions and deductions learned at school and developing to further sophistication at the university level. With commitment and effort, learning skills can continue to grow to full maturity as one works toward the frontiers of mathematical research.

Tall takes a structuralism position: that there are fundamental mental concepts that are the scaffolds of our mental processes, and that there are successive levels of mathematical thinking that support advances in quantitative as well as spatial understanding. He presents learning as part of sensory-motor experiences connected to our communication facility of language and symbolism. First, there are the physical objects that are experienced and categorized both visually and tactilely. Operations are performed on those physical objects. In chapter 4 he introduces what he calls "set-befores" and "met-befores", frameworks that he uses to describe how we move from one stage to another. We are all born with "set-before" structures that slowly make neural connections generated by our interactions with the world. They permit us to move forward in understanding the world before we get to "met-befores".

The brain comes equipped with the ability to recognize and track small numbers of objects, a quantity sense that uses no learned counting rules. We come equipped with knowing what is heavy and what is light, which way is up and which is down. Animals and birds are born with such equipment. But Tall hypothesizes that human set-before thinking has three components—recognition, repetition, and language. All animals and birds can sense and recognize patterns, similarities, and differences. The chickadee knows how many eggs it laid in its nest. If one is missing, it knows that without counting. It is also equipped with very complex set-befores, and uses them to instinctively build its nest without ever being taught how. Humans repeat complex actions as a way to internalize thinking and perform those actions automatically. We learn our alphabet and our multiplication table that way. On page 85 Tall writes, "It is the basis of *procedural* thinking, where procedures can be performed automatically without being conscious of the details, except where particular decisions need to be made at critical points of the operation."

But the essential component of our set-befores is language. According to Tall, language, along with recognition and repetition elevates mathematical thinking, extending imagination to understanding concepts beyond what we see repeated in our physical existence. He gives, as example, the notion of potential infinity. We can repeat an operation over and over again until we see the succession carry on to its extreme, or to compartmentalize it to an understanding of its abstraction. We start with a number, move on to the next. We soon realize that we will always have a next number, and that all the numbers we can count might be categorized by naming the collection. We might call the category *the natural numbers*.

Then come the met-befores, those interpretations of new experiences that come from all those experiences—positive, negative, and problematic—we have met before. This is a very tricky concept because we are told on page 88 that it is not the experience itself, but rather what "it leaves in the mind that affects our current thinking." The subtlety here is that experiences have meta-lives that incubate and develop into mental paradigms that control our persuasions and suspicions of truth. Tall takes the example of i, the complex number whose square is -1. When we first encounter it in our studies, we recall that squares are never negative. That jars with a met-before that comes from our experience with real numbers: squares of real numbers are always positive. The moment we encounter a strange number, such as $i = \sqrt{-1}$, we make corroborating adjustments to our old met-befores. Similarly, the product of two whole numbers is bigger than either one, but that is not the case for fractions. Although we know that long-term development depends on preparedness and of making sense of each successive level of experience, not every new level of experience is supportive of the next. Some may be problematical. So the big issue for learning mathematics is not just future learning but how we intend to surmount earlier experiences that might be problematic for new situations and advancement.

Tall's work applies to experts as well as students. History of science tells us that solidly ensconced viewpoints are known to impede seeing new ways of meeting problems. I'm not sure how this can be proved through modern research experiments on how we learn to think, but we do have plenty of evidence from Thomas Kuhn and James Bryan Conant.

These met-befores are central to the book and to how humans learn to think mathematically. They create the template for moving mathematical reasoning further. The subtitle is *Exploring the Three Worlds of Mathematics*. What are those three worlds? Mathematics has its divisions, such as Algebra, Analysis, Geometry, and perhaps it could be broken down further, but Tall's worlds are about three different thinking developments. Tall calls his three worlds *conceptual embodiment*, *operational symbolism*, and *axiomatic formalism*. They each grow mature from a sensori-motor-linguistic experience encouraged by the set-befores of recognition, repetition and language, as well as by the met-befores of earlier experiences. Briefly, the first (the conceptual) envelops all that we do to perceive mental images to enhance emergent imagination. The core example is that from some counting numbers we verbalize some mental images, and by repetition we form increasingly sophisticated understandings of the concept of number itself. The second (the operational) is the world of all those physical actions that get us to understand how to proceed with mathematical operations. Performing procedural calculations on a symbolic and categorical level compresses mathematical experiences into presences of thinkable concepts. The third is the construction of formal or abstract knowledge set through axioms and definitions. Though the worlds are distinct, they are integrated. So, for an example of how a person learns how to think mathematically, we might look at the maturation of a child as her thinking becomes more sophisticated. She starts by exploring shapes and spaces, forms concepts of number from counting objects. Eventually she comes to use definitions, deduce properties, and form abstractions. From embodiment and symbolism she develops mathematical reasoning skills, and ultimately all the consequences of experimenting with familiar objects and operations that collaborate to form a sophisticated mathematical reasoning structure. These notions are expanded and explained with excellent examples as chapter 6 plows through a minefield of educational psychology jargon to form a better sense of what Tall means by the three worlds of mathematics.

Early on, Tall introduces his notion of a "crystalline concept". On page 27 he defines it as a "thinkable concept that has a necessary structure as a consequence of its context." For an example, he takes the set of whole numbers to be a crystalline concept, because it is a concept that is built in the mind through contextual experiences hinted by the way the collection of numbers add, subtract, multiply and divide. It is a multifaceted concept that employs bordering conceptual notions that are continually being modified. At very early stages, young children use relationships between numbers to simplify and *crystalize* their understanding of arithmetic as a concept. For a geometric example he takes the Euclidean triangle. We have a concept of the triangle that is crystalline; if it has two equal sides it must have two equal angles. In its more common use crystalline means the structural order of atoms and molecules in a solid. So the extension of the word, from physical chemistry to thinkable concepts, works well. The crystalline concept develops into knowledge structures all along the learning path, mostly from real world applications, but also from reflecting and imagining in formal thinking. When mathematical structures are compressed into thinkable structures, they merge with other knowledge assemblies that crystalize into more sophisticated mathematical structures.

On page 171 Tall tells us that “As practical mathematics shifts to theoretical mathematics, it moves away from the familiar ideas in the everyday world to new contexts that involve new embodiments with their own sense of structure and operations that may be symbolized in new ways. But once the basic ideas of symbolism in the new context become familiar, the relationships between the symbols may take on a life of their own to give enormous power of calculation and manipulation that far exceeds the capacity of the initial embodied idea.”

The book is divided into four sections. Section I is a persuading preamble to the book. Section II looks at the foundations of mathematical thinking, starting with the question of how young children build their ideas of space, shape and number. It begins with an introduction of relevant terms for a theory of learning centered on set-befores and met-befores, and follows a mathematical maturation line that ends with how experts might organize formal proofs.

The third part gives us a concise historical evolution of mathematics condensed to one chapter, starting with the Babylonians and Egyptians and ending with the modern computer and its influence on pre-computer days of thinking. More than just a tool that gives us static visual output, the computer provides a means for the student or investigator to control variables in ways that simulate sophisticated interactive experiences. Tall puts it this way on page 249: “The balance between concept and process changes the human focusing more on the concept and the computer taking care of the process.” It’s more than just a short history of mathematics. In just 36 pages, we get a rich picture of the conceptual transitions, shifts in ways of thinking. The chapter points to moments when one way of thinking becomes an obstruction to advancement. Think of the moment when the square root of a negative number was finally admitted to the number world and damagingly called *imaginary*.

The last part takes us into university mathematics and beyond, starting with how ideas of calculus bring us to think of infinity differently and ending with advanced mathematical research. On page 402 Tall asks, "Of what value is this general theory to individuals playing a particular role in the teaching and learning of mathematics? What does it have to say to teachers of young children, to university mathematicians, to curriculum designers, to theorists in various communities of practice, or to learners themselves?" The answers are there.

Throughout the book there are schematic flow illustrations. Some are quite helpful in encapsulating the ideas of the text. They are packed with information on how the learning moves from one stage to another. My first reaction was confusion. The illustrations seemed to be more complex than needed and not helping conceptions of the textual material itself. However, after looking more carefully I found them to be very helpful. Some are more useful than others and some downright confusing. Perhaps I was not understanding the schematic representations.

This is a book that should be assigned reading for all K-12 teachers of mathematics, and perhaps administrators who might have some influence on changes in curriculum. The hope is that it will lead to better understanding of particular sensitivities to both teaching and learning styles. It is a book that encapsulates almost a half-century of important work in the field of mathematics education. Tall's book is not about how humans think, but rather about how humans *learn to* think. The book is filled with case stories from an ample range of research accounts. The scope is as wide as it is broad: from how young children learn to count to how research mathematicians deal with obstructions. One excellent feature of this book is that it does not avoid the problematic ideas that are part of the story of learning. Tall spends a great many pages focused on problematic ideas. In chapter 4 he gives a numerous examples useful to early learning that prove to be obstructions to new situations. In the 1960s there were a few people working on theories of how mathematical thinking develops. Research on the topic continued into the 1990s. But Tall's work is the crystallization of all previous concepts and one need look no further for a conceptual sketch of the field.

How Humans Learn to Think Mathematically is written in the style intended for a popular audience. It is an easy read. The language is concise and understandable, though the depth is vast. There is a repetition, a certain spiraling of information that I am sure Tall consciously built into the book. I found myself applauding that repetition after realizing that it was bringing his readers up through the scaffolds of mental processes by the same successive stages of thinking that he was writing about. Each spiraling repetition lifted me to the next structural level of understanding.

References

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